Faster Algorithms for Sparse Fourier Transform

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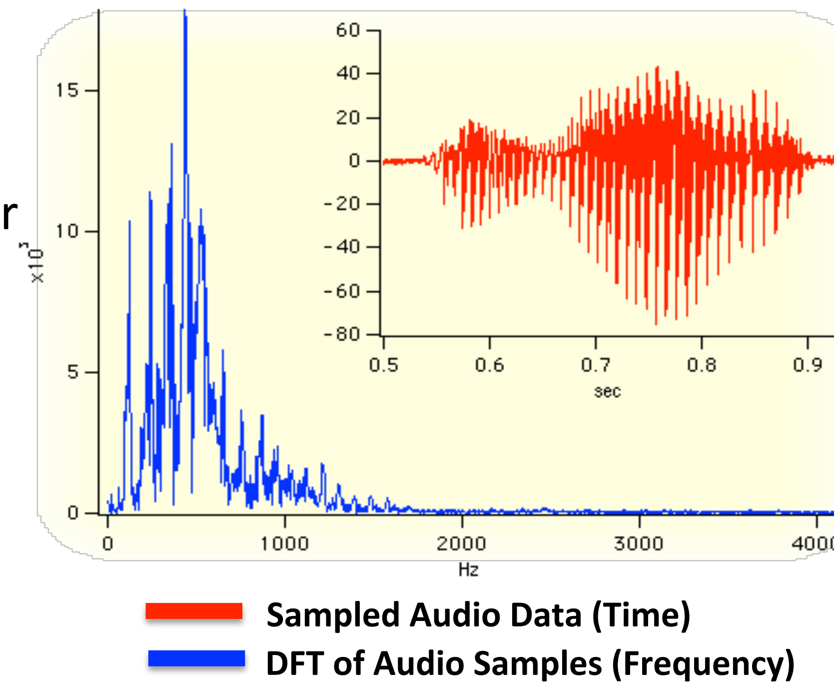
Material from:

- Hassanieh, Indyk, Katabi, Price, “Simple and Practical Algorithms for Sparse Fourier Transform, SODA’12.
- Hassanieh, Indyk, Katabi, Price, “Nearly Optimal Sparse Fourier Transform”, STOC’12.
- Hassanieh, Adib, Katabi, Indyk, "Faster GPS Via the Sparse Fourier Transform”, MOBICOM’12
- Ghazi, Hassanieh, Indyk, Katabi, Price, Lixin, “Sample-Optimal Average-Case Sparse Fourier Transform in 2D

Fourier Transform

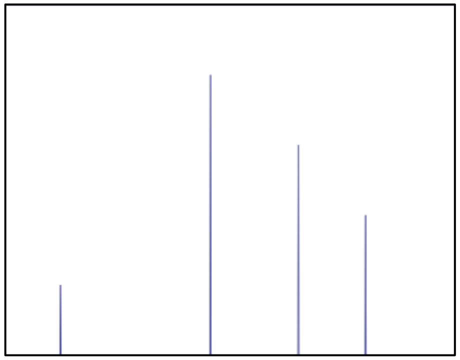
- Discrete Fourier Transform:
  - Given: a signal x[1…n]
  - Goal: compute the frequency vector x' where

$$x'_f = \Sigma_t \, x_t \, e^{-2\pi i \, tf/n}$$



- Very useful tool:
  - Compression (audio, image, video)
  - Signal processing
  - Data analysis
  - Communication
  - Computation (convolution, error-correcting codes, ..)
  - …

Known algorithms



- Fast Fourier Transform (FFT) computes the frequencies in time $O(n \log n)$
- But, we can do better if we only care about small number k of "dominant frequencies"
  - E.g., recover assume it is k-sparse (only k non-zero entries)
- Algorithms:
  - Boolean cube (Hadamard Transform): [KM] (cf. [GL])
  - Complex FT: [Mansour'92, GGIMS'02, AGS'03, GMS'05, Iwen'10, Akavia'10]
- Best running time*: $k \log^c n$ for some $c=O(1)$ [GMS05, Iwen'10]
  - Improve over FFT for $n/k >> \log^{c-1} n$
  - In fact, the running time can be sub-linear in n
- Problem:
  - c is around 4
  - Need $n/k > 40{,}000$ to beat FFTW for $n=2^{22}$
- Goal:
  - Theory: improve over FFT for all values of $k=o(n)$
  - Improve in practice

*Assuming entries of x are integers with $O(\log n)$ bits of precision.

Our results: theory

- All algorithms randomized, with constant probability of success, n is a power of 2
- Exactly k-sparse case : $O(k \log n)$
  - Optimal if FFT optimal for $k>n^{\Omega(1)}$
- Approximately k-sparse case - l_2/l_2 guarantee:
  $||x'-y'||_2 \leq C \min_{k\text{-sparse } z'} ||x'-z'||_2$ for an approx $C>1$
  - We get $O(k \log(n) \log(n/k))$ time
  - Improves over FFT for any $k << n$
- Slower (but sub-linear) algorithm for a stronger l_∞/l_2 guarantee
- Same time, sample complexity reduced by log n factor (i.e., to $O(k)$* or $O(k \log(n))$), for the average case in 2D
  - Sample optimal (even in the average case)

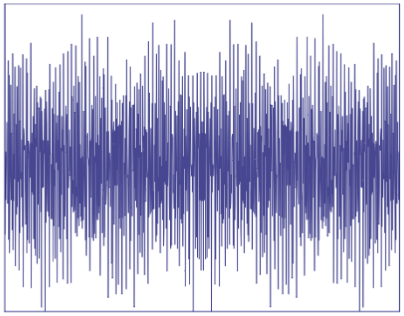
*Similar result was recently independently discovered by Pawar and Ramchandran

Our results: experiments

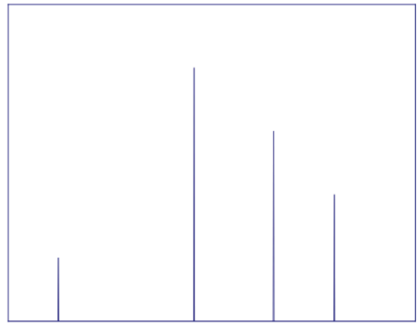
- Significant improvement in running times over prior work
  - E.g., for $n=2^{22}$, a variant of our algorithm (for the exactly k-sparse case) is faster than FFTW for k up to about 2^{17}
  - Best prior implementation of [GMS'05] due to Iwen achieved this breakpoint for k up to about 2^{7}
- Applications:
  - GPS synchronization [Hassanieh-Adib-Katabi-Indyk'12]
  - Spectrum sensing [Yenduri-Gilbert'12], [Hassanieh-Shi-Abari-Hamed-Katabi'13]
  - Magnetic Resonance Spectroscopy [Shi-Andronesi- Hassanieh-Ghazi-Katabi-Adalsteinsson'13]
  - Exploiting Sparseness in Speech for Fast Acoustic Feature Extraction [Nirjon-Dickerson- Stankovic-Shen-Jiang'13]

Sparse FFT – exact sparsity

Intuition: Fourier



Time Domain Signal

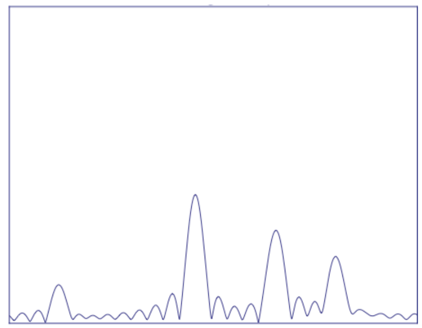


Frequency Domain

n-point DFT : $n\log(n)$

$\mathbf{x} \rightarrow \mathbf{x}'$

Cut off Time signal

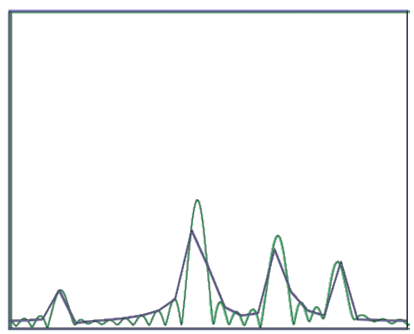


Frequency Domain

n-point DFT of first B terms : $n\log(n)$

$\mathbf{x}\times$ **Boxcar** $\rightarrow$ $\mathbf{x}' *$ **sinc**

First B samples



Frequency Domain

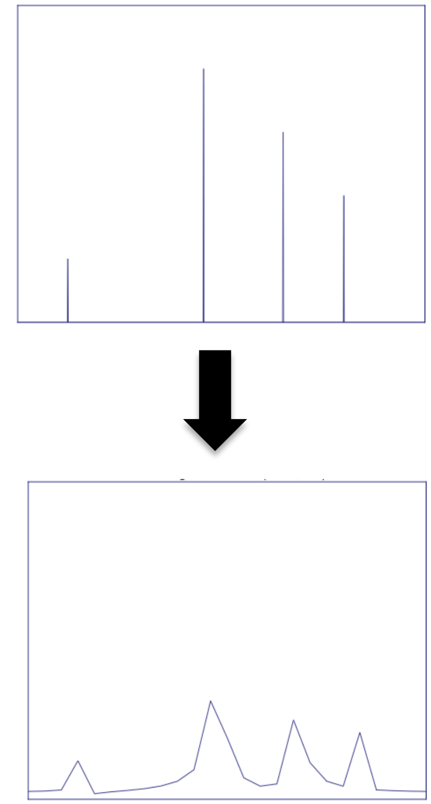
B-point DFT of first B terms: $B\log(B)$

Alias $(\mathbf{x}\times \mathit{Boxcar})$

$\downarrow$

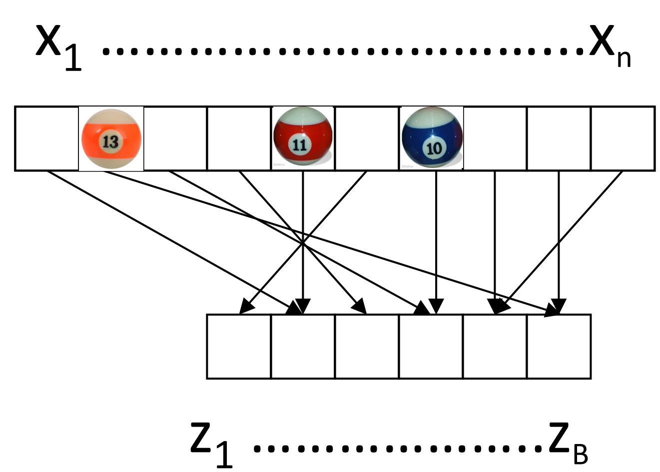
Subsample $(\mathbf{x}' * sinc)$

Balls and bins



- We we would like this

... to act like a balls and bins process:

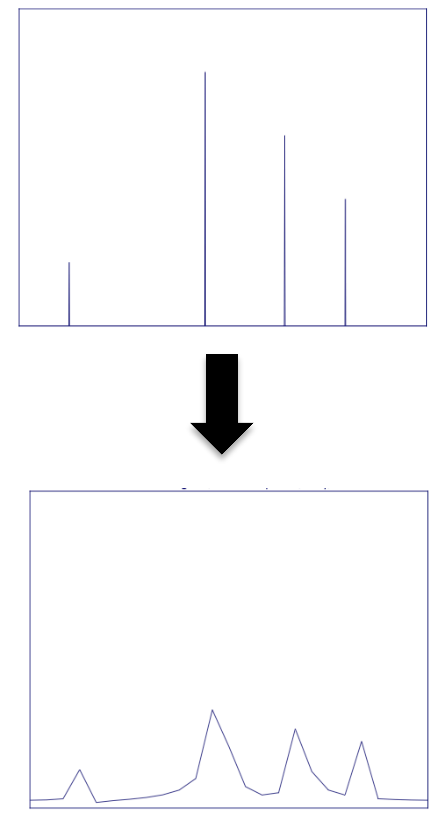


- Each non-zero coefficient is "hashed" into one of B bins
- Coefficients in the same bin sum up
- Most coefficients are isolated in a bin so they can be easily* recovered

*Charikar-Chen-FarachColton'02, Estan-Varghese'03, Cormode-Muthukrishnan'04, Gilbert-Strauss-Vershynin-Tropp'06, Berinde-Gilbert-Indyk-Karloff-Strauss'08, Sarvotham-Baron-Baraniuk'06,'08, , Lu-Montanari- Prabhakar'08, Wang-Wainwright-Ramchandran'10, Akcakaya-Tarokh'11….

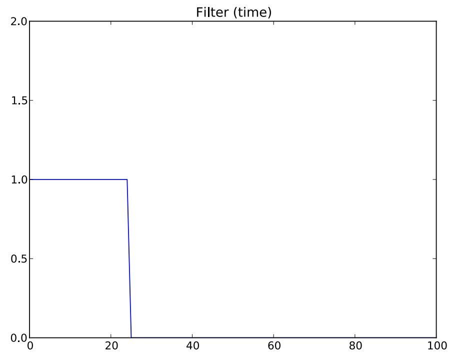
Towards balls and bins

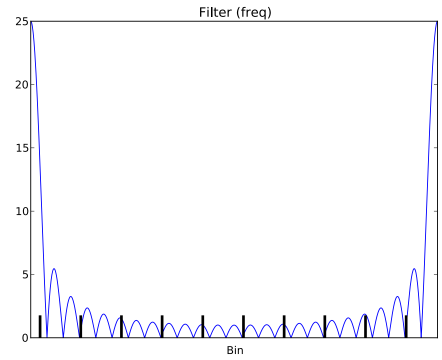
- Issues:
  - “Hashing”: needs a random hashing of the spectrum
  - “Leaky” buckets
  - Finding the support



Reducing leakage

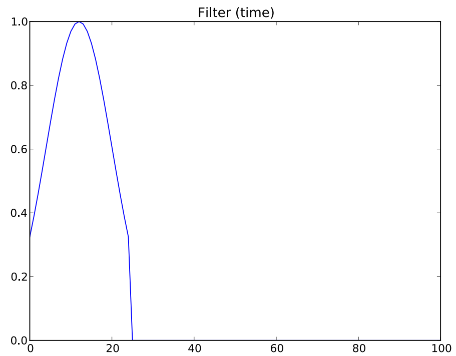
Filters: rectangular filter (used in[GGIMS02,GMS05])

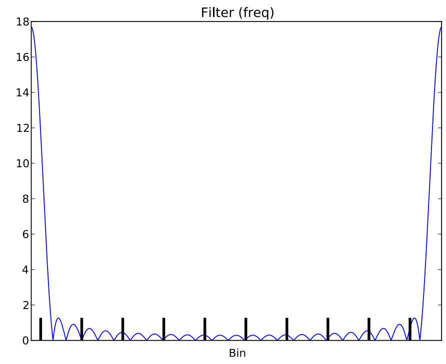




- Rectangular -> Sinc
  - Polynomial decay
  - Leaking many buckets

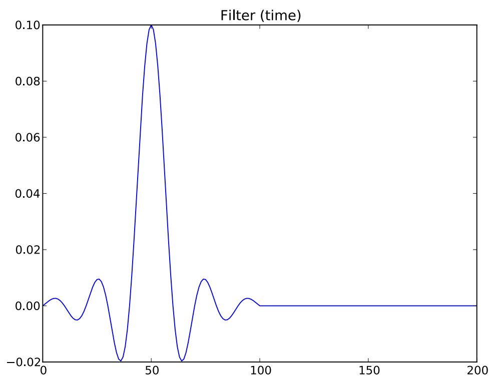
Filters: Gaussian

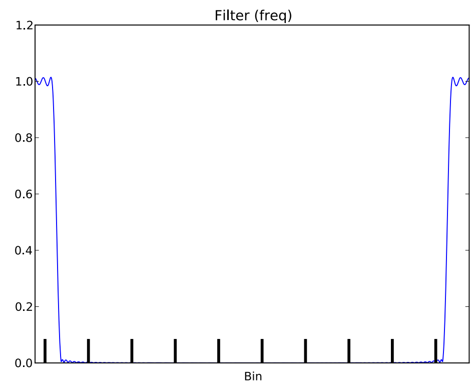




- Gaussian -> Gaussian
  - Exponential decay
  - Leaking to $(\log n)^{1/2}$ buckets

Filters: Sinc × Gaussian

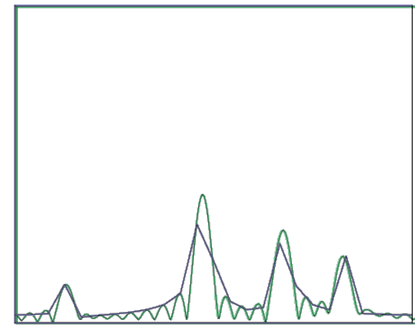




- Sinc × Gaussian -> Boxcar*Gaussian
  - Still exponential decay
  - Leaking to <1 buckets
  - Sufficient contribution to the correct bucket
- Actually we use Dolph-Chebyshev filters

Finding the support

Finding the support



- $y' =$ B-point DFT $(x \times F)$

  $= \text{Subsample}(x' * F')$

- Assume no collisions:
  - At most one large frequency hashes into each bucket.
  - Large frequency f_1 hashes to bucket b_1

    $y'_{b1} = x'_{f1} F'_{\Delta} +$ ~~leakage~~

- Let x^{τ} be the signal time-shifted by τ, i.e. $x^{\tau}_{t} = x_{t-\tau}$
- Recall $\text{DFT}(x^{\tau})_f = x'_f \, e^{-2\pi i \tau f/n}$
- $y^{\tau}{}' =$ B-point DFT $(x^{\tau} \times F)$

  $y^{\tau}{}'_{b1} = x'_{f1} e^{-2\pi i \tau f1/n} F'_{\Delta} +$ ~~leakage~~

Finding the support, ctd

- At most one non-zero frequency f_1 per bucket b_1
- We have

$$y'_{b1}=x'_{f1}F'_{\Delta}$$

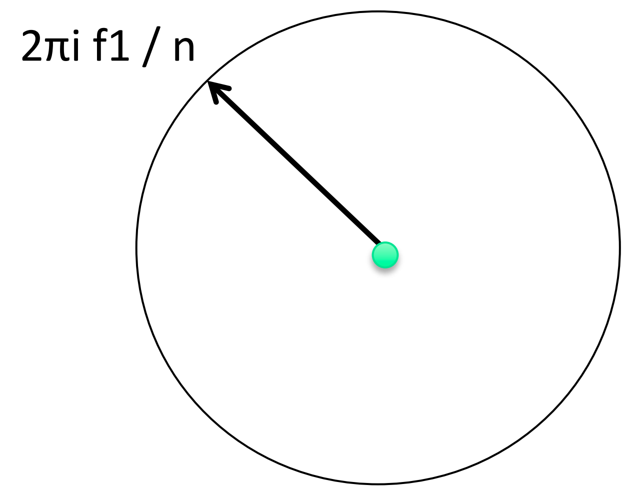
and

$$y'^{\tau}_{b1}=x'_{f1}e^{-2\pi i\,\tau\,f1/n}\,F'_{\Delta}$$

- So, for $\tau=1$ we have

$$y'_{b1}/y'^{1}_{b1} = e^{-2\pi i\,f1\,/n}$$

- Can get f1 from the phase



Spectrum Hashing (used in[GGIMS02,GMS05])

- Every iteration needs new random hashing:
  - Permute time domain signal → permute frequency domain
  - Let

$$z_t = x_{\sigma t} \, e^{-2\pi i \, t \, \beta/n}$$

  - If σ is invertible mod n

$$z'_f = x'_{1/\sigma \, f+\beta}$$

Algorithm

(exactly k-sparse case)

- Iteration i :
  1. Set the number of buckets $B_i \approx k/2^{i-1}$
  2. Permute spectrum : $z_t = x_{\sigma t}\, e^{-2\pi i f \beta/n}$
  3. $y' = B_i$ -point DFT ($z \times F$) = Subsample($z' * F'$)
  4. Repeat with time shift to get y'^{τ}
  5. Subtract large frequencies recovered in previous iterations
  6. Recover locations and values of remaining large frequencies
- Iteration i recovers $k/2^{i-1}$ of the large frequencies with probability 3/4 in $O(B_i \log n)$ time
- Total time $O(k \log n)$
  - Steps 3,4 dominated by $B_1=k$
  - Step 5 takes $O(k)$ time per each of the $O(\log n)$ iterations

Future directions

Question 1: Sample complexity

Algorithm	Time	Samples	Lower bound
SFFT 3.0 (exact)	O(k log n)	O(k log n)	O(k)
SFFT 4.0 (compressible)	O(k log(n) log(n/k))	O(k log(n) log(n/k))	O(k log (n/k))

- Can match the lower bound for average-case sparsity [Ghazi, Hassanieh, Indyk, Katabi, Price, Lixin'13; Pawar, Ramchandran'13]
- Optimality in the worst-case ?

Question 2: Higher dimension

- The higher dimension, the sparser the data
- Alas, in d-dimension, the complexity is $O(k (\log n)^{d+1})$
- Question: Improve to $O(k \log(n^{d+1}))$?

Question 3: Uniform bounds

- Suppose we would like a sampling pattern that works for all x
- By [Candes-Tao, Rudelson-Vershynin] we know that $O(k \log^4 n)$ samples suffice
  - However, the recovery time is npolylog n (e.g., CoSaMP)
- Fastest deterministic sub-linear time algorithm has k^2 polylog n complexity [Iwen]
  - Mimics the bounds achievable for RIP using sparse matrices
- Question: can we get k^{2-a} polylog n bound for some a>0 ?

Conclusions

- O(k log n) times/samples achievable for the k-sparse case
- O(k log n log(n/k)) achievable for the L2/L2 guarantee
- Questions:
  - Fewer samples (worst case)
  - Higher dimensions
  - Uniform